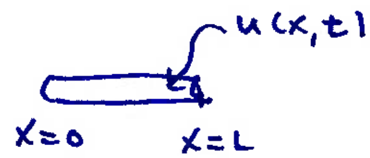
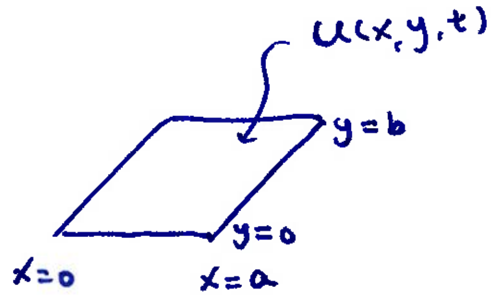
Higher-Dimension PDE

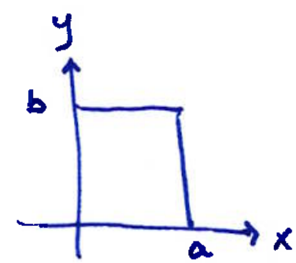
1-D $\quad u_t = k u_{xx}$



2-D $\quad u_t = k(u_{xx} + u_{yy})$



today: edges are kept at $u = 0$



$$u_t = k(u_{xx} + u_{yy}) \qquad 0 < x < a \qquad 0 < y < b$$

BCs: $u(x, 0, t) = 0$ (bottom)

$u(a, y, t) = 0$ (right)

$u(x, b, t) = 0$ (top)

$u(0, y, t) = 0$ (left)

IC: $u(x, y, 0) = f(x, y)$

same method: separation of variables

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

rewrite heat eq: $u_t = k(u_{xx} + u_{yy})$

$$XYT' = k(X''YT + XY''T)$$

$$\vdots$$

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T'}{kT} = \text{constant} = -\lambda$$ same separation constant

$\frac{X''}{X}$: depends on x alone

$-\frac{Y''}{Y} + \frac{T'}{kT}$: depends on y and t

first ODE: $\boxed{X'' + \lambda X = 0}$

$$-\frac{Y''}{Y} + \frac{T'}{kT} = \text{constant} = -\lambda$$

rewrite: $\frac{Y''}{Y} = \frac{T'}{kT} + \lambda = \text{constant} = -\mu$ (another constant) from that we get two more ODEs

$$\boxed{Y'' + \mu Y = 0}$$ looks just like X equation

$$\boxed{T' + k(\mu+\lambda)T = 0}$$

BCs: $u(x,0,t) = 0 \longrightarrow Y(0) = 0$

$u(x,b,t) = 0 \longrightarrow Y(b) = 0$

$u(0,y,t) = 0 \longrightarrow X(0) = 0$

$u(a,y,t) = 0 \longrightarrow X(a) = 0$

$X'' + \lambda X = 0, \quad X(0) = X(a) = 0$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{a^2}} \quad n = 1, 2, 3, \ldots$$

$$\boxed{X_n = \sin\left(\frac{n\pi}{a}x\right)}$$

$Y'' + \mu Y = 0, \quad Y(0) = Y(b) = 0$

$$\boxed{\mu_m = \frac{m^2\pi^2}{b^2}} \quad \boxed{Y_m = \sin\left(\frac{m\pi}{b}y\right)} \quad m = 1, 2, 3, \ldots$$

$$T' + k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)T = 0$$

$$\boxed{T_{mn} = e^{-k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)t}}$$

for $n = 1, 2, 3, \ldots$
$m = 1, 2, 3, \ldots$

for each (m, n) pair, there is one solution

$$u_{mn} = T_{mn} X_n Y_m$$

general solution: sum over n and m

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)t} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)}$$

IC: $u(x, y, 0) = f(x, y)$ initial heat distribution

$$f(x,y) = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{a}\right) \right] \sin\left(\frac{m\pi y}{b}\right)$$

double sine series

"constant" if
x is fixed → call it C

if x is fixed, it looks like

$$\underset{(x \text{ fixed})}{f(x,y)} = \sum_{m=1}^{\infty} C \sin\left(\frac{m\pi y}{b}\right)$$

"regular" sine series

$$C = \underbrace{\frac{2}{b} \int_0^b f(x,y) \sin\left(\frac{m\pi y}{b}\right) dy}_{\text{function of } x} = \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{a}\right)$$

↑ constant

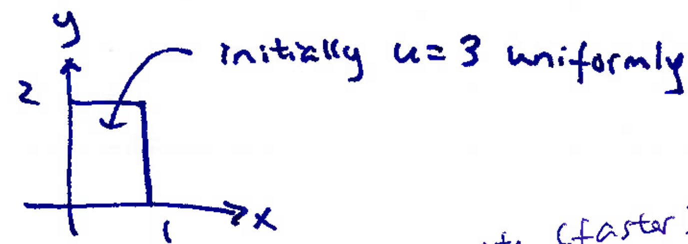
"regular" sine series

$$A_{mn} = \frac{2}{a} \int_0^a \left[\frac{2}{b} \int_0^b f(x,y) \sin\left(\frac{m\pi y}{b}\right) dy \right] \sin\left(\frac{n\pi x}{a}\right) dx$$

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi x}{a}\right) dy\, dx$$

<u>example</u> $a=1,\ b=2,\ f(x,y)=3$

$k=1$

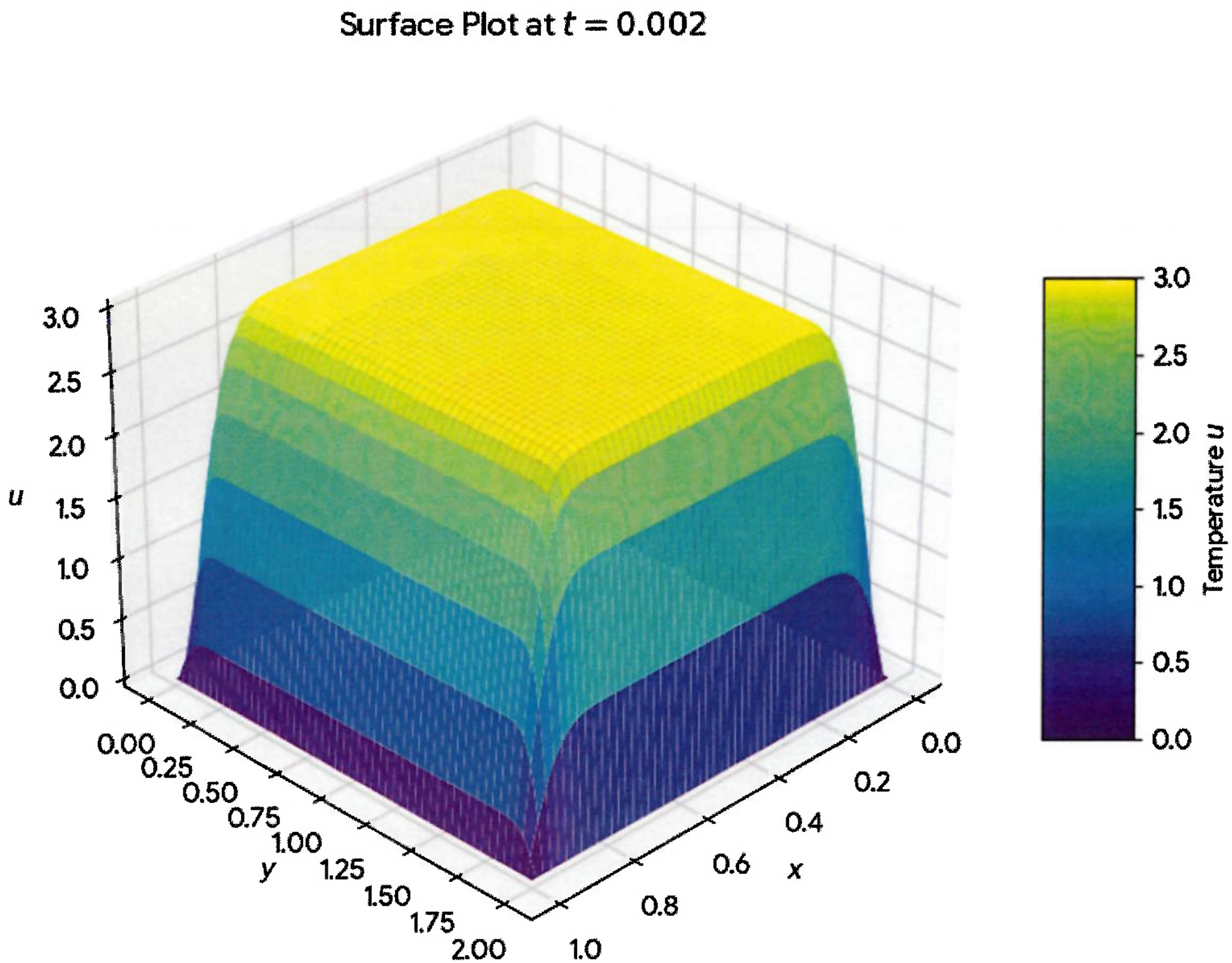


⋮

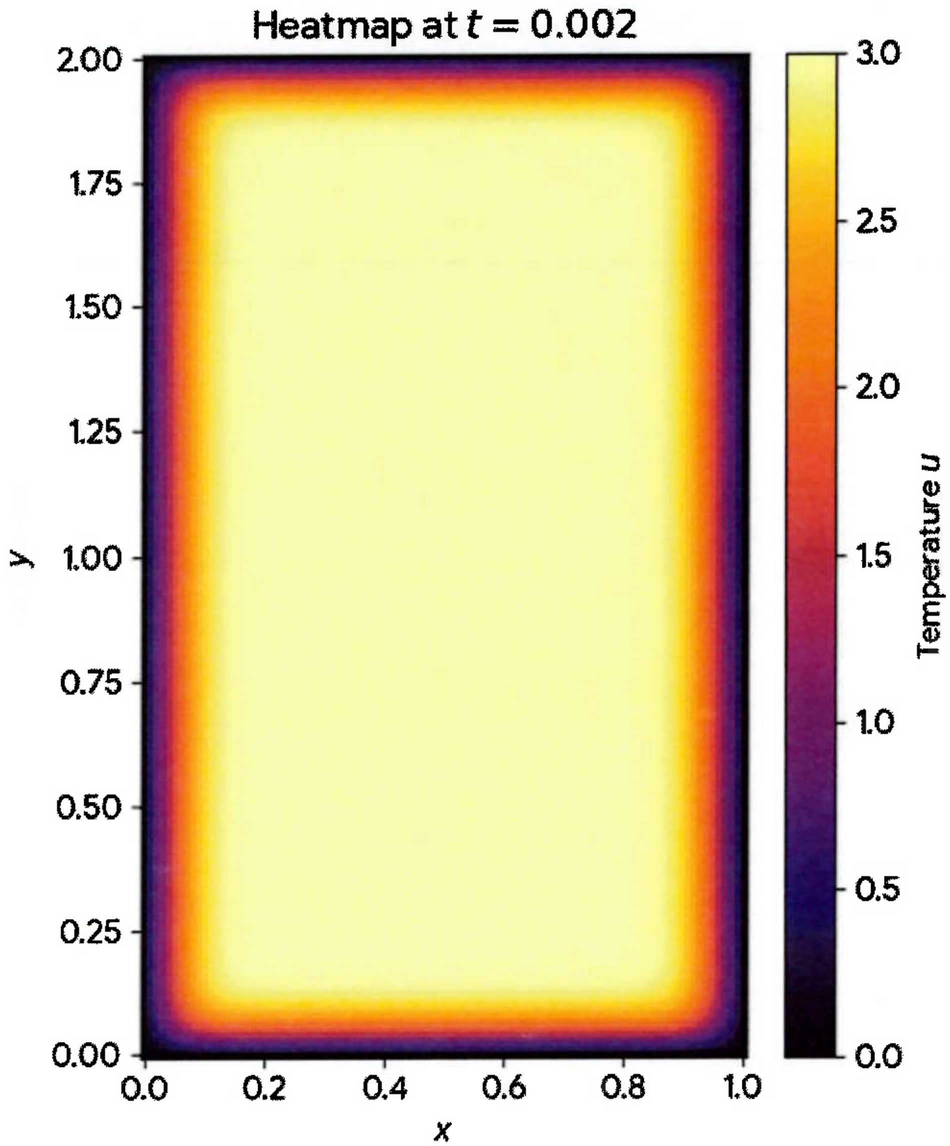
x decay rate (faster)

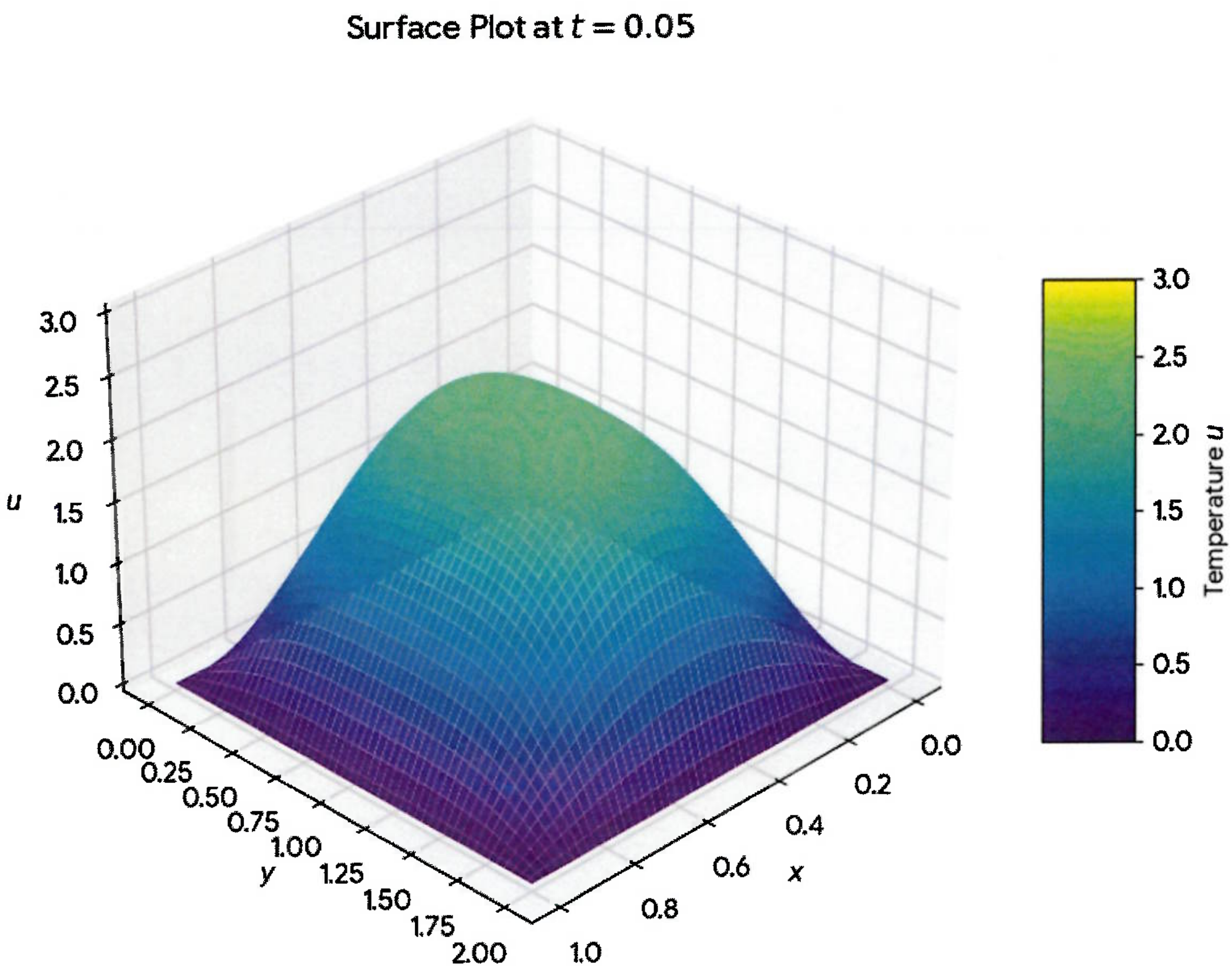
y decay rate

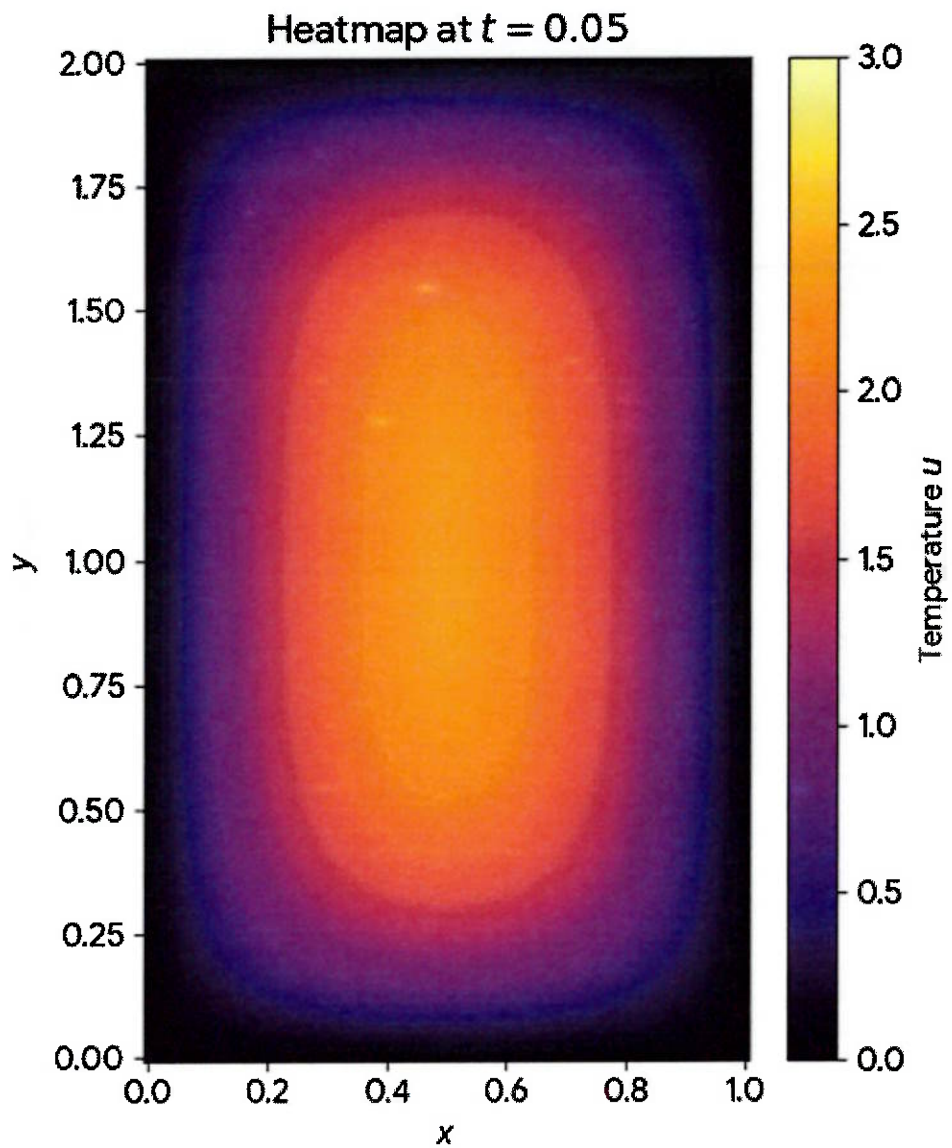
$$u(x,y,t) = \sum_{m\ \text{odd}}^{\infty} \sum_{n\ \text{odd}}^{\infty} \frac{48}{mn\pi^2} \sin(n\pi x) \sin\left(\frac{m\pi y}{2}\right) e^{-\left(n^2\pi^2 + \frac{m^2\pi^2}{4}\right)t}$$

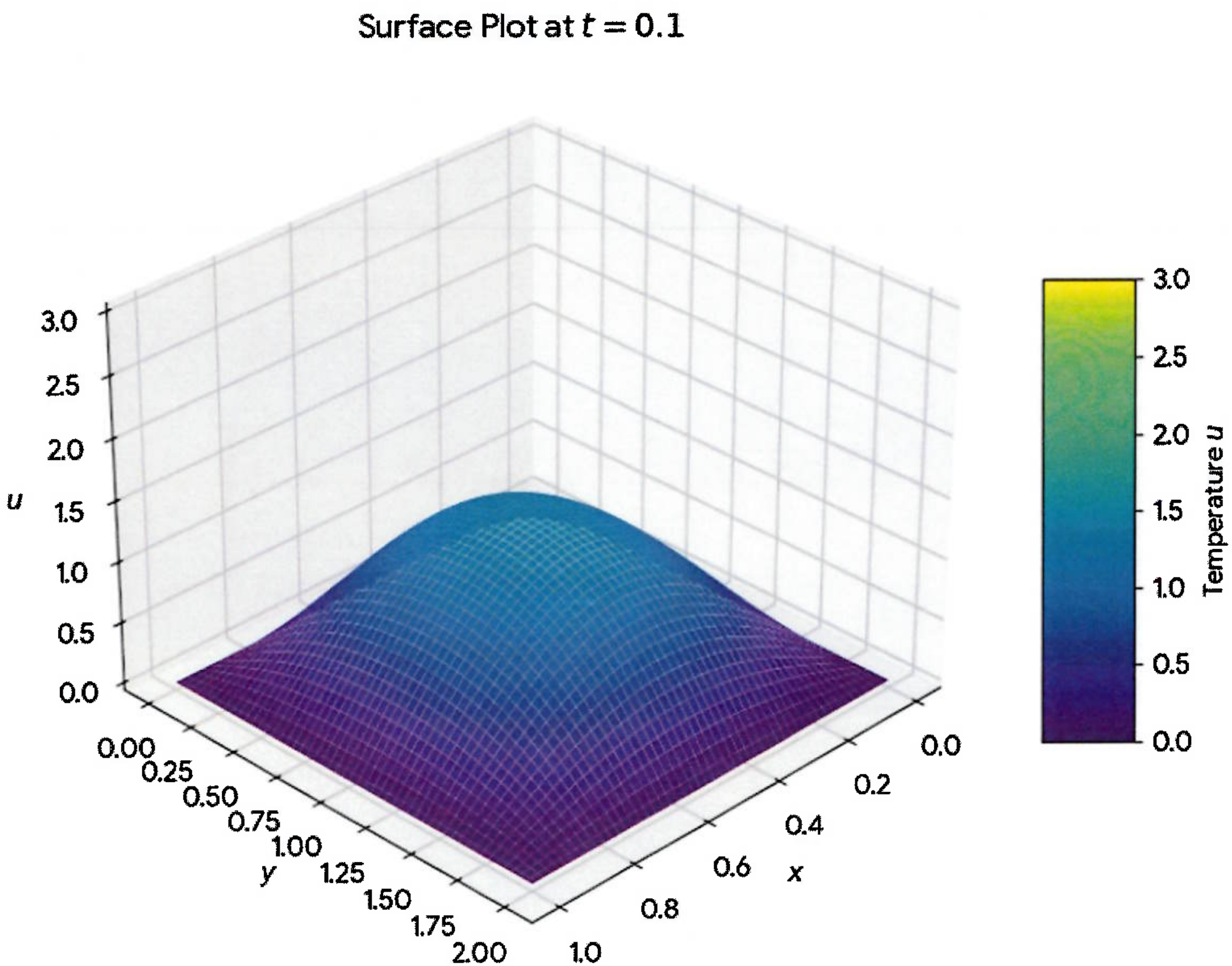


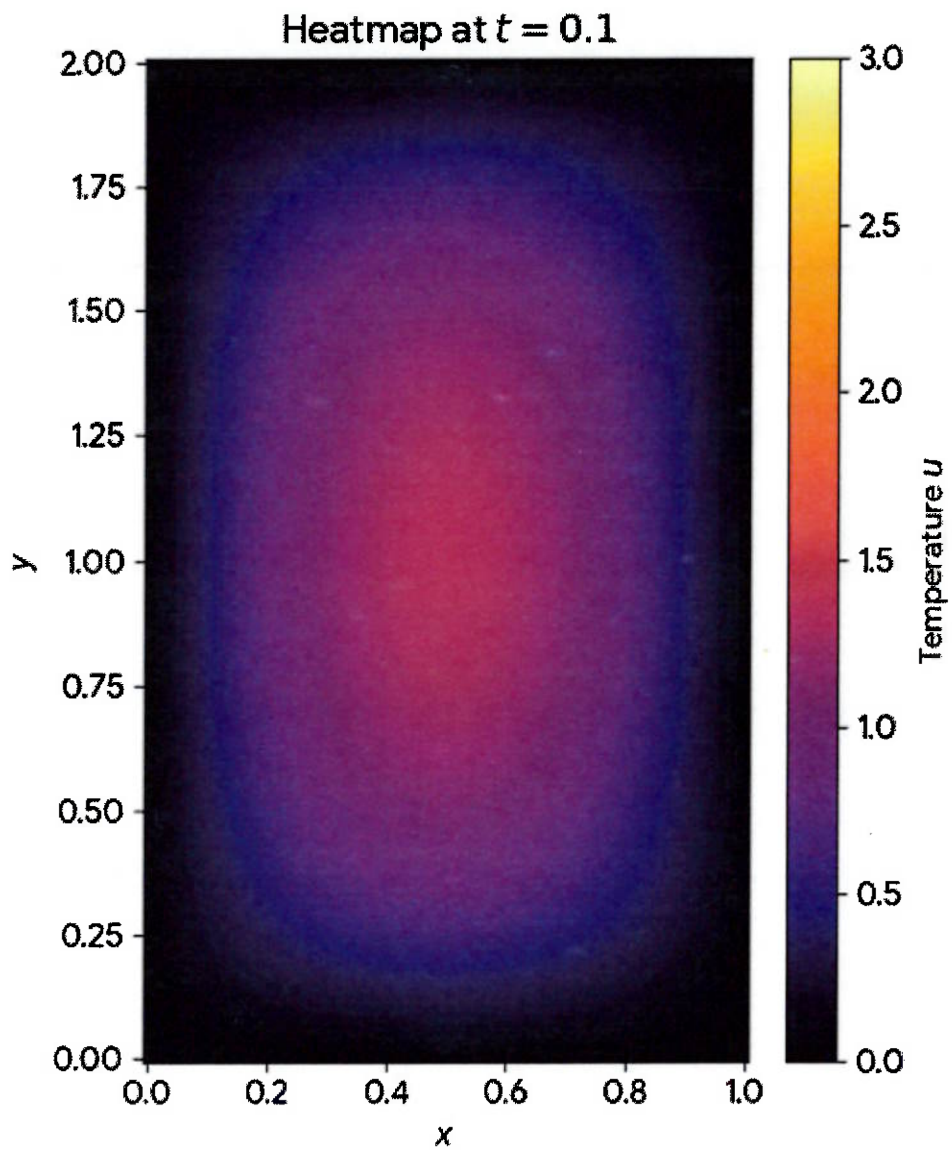
Surface Plot at $t = 0.002$
u
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x
0.0
0.2
0.4
0.6
0.8
1.0
Temperature u
3.0
2.5
2.0
1.5
1.0
0.5
0.0


Heatmap at $t = 0.002$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u

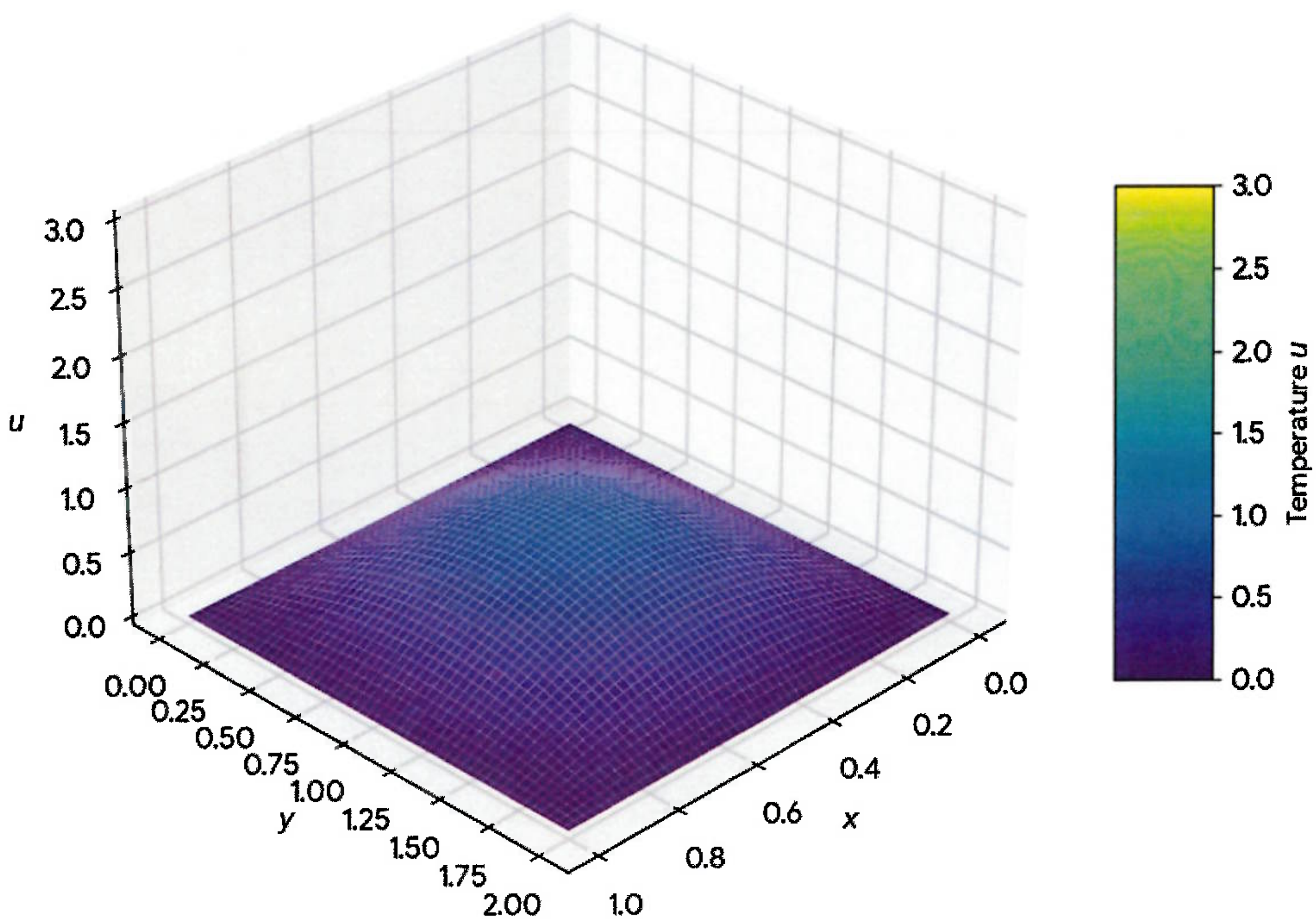

Surface Plot at $t = 0.05$
u
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x
0.0
0.2
0.4
0.6
0.8
1.0
Temperature u
3.0
2.5
2.0
1.5
1.0
0.5
0.0

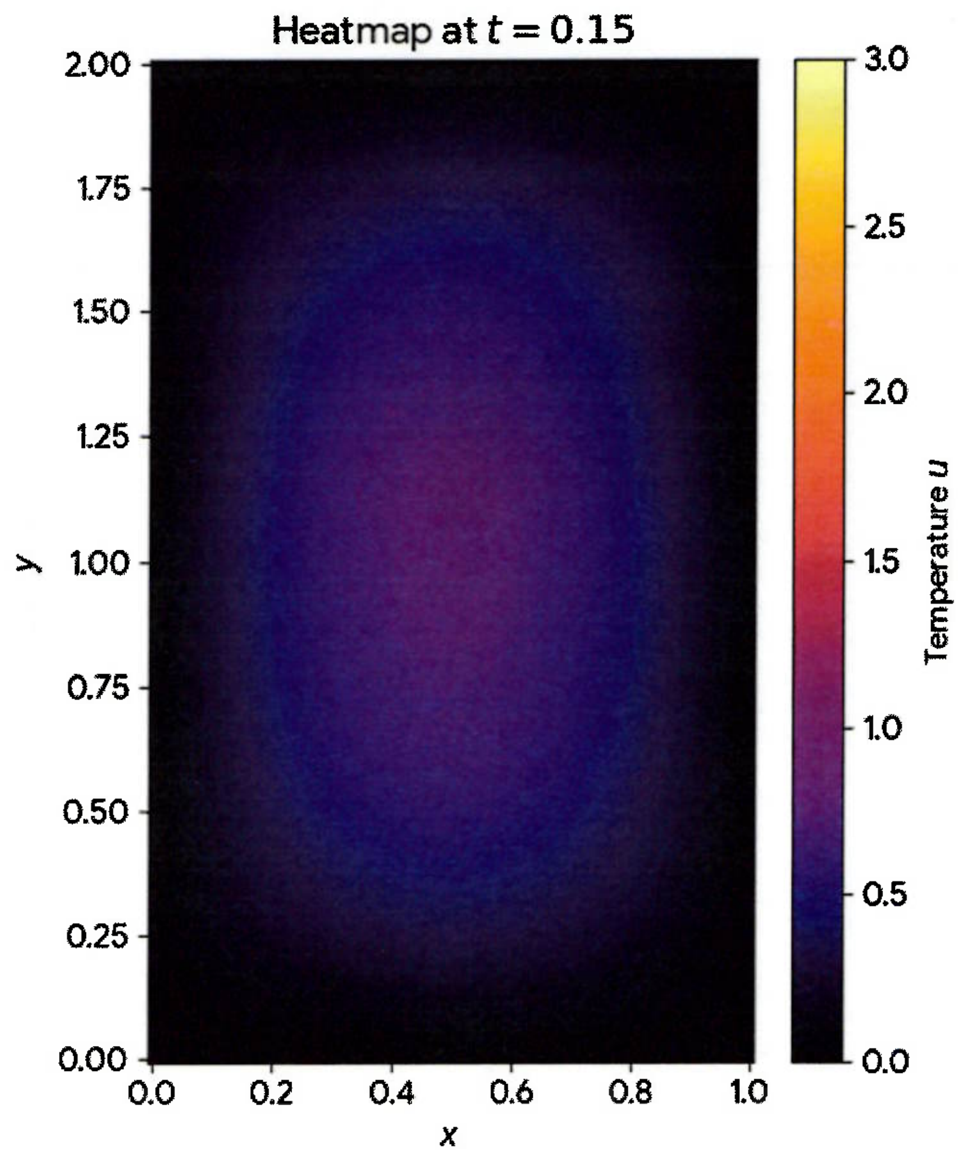

Heatmap at $t = 0.05$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u


Surface Plot at $t = 0.1$
u
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x
0.0
0.2
0.4
0.6
0.8
1.0
Temperature u
3.0
2.5
2.0
1.5
1.0
0.5
0.0


Heatmap at $t = 0.1$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u

Surface Plot at $t = 0.15$





Heatmap at $t = 0.15$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u